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Analysis of compressible flows around a uniformly expanding circular cylinder and sphere

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The flows caused by a uniformly expanding circular cylinder or sphere in a perfect gas at rest at various (from small to very large) expansion velocities are analysed by a method of successive approximations in which the first approximation represents the incompressible flow. The shock waves which form around the bodies are also treated. The results for a sphere, even up to the third approximation, agree closely with Taylor's (1946) calculations for all the expansion rates.

1. Introduction

Similarity flows of a perfect gas are of practical importance in applications to steady supersonic or hypersonic flows around slender bodies and many such investigations have been performed. In particular, the flows around pointed conical bodies, i.e. conical flows, which are analogous to two-dimensional piston problems, have been studied analytically and numerically. While, for the piston problem, Taylor (1946) solved the flow around a uniformly expanding sphere, Sedov (1959) obtained the general solutions for a sphere and a circular cylinder numerically.

In this paper the similarity flows caused by a uniformly expanding circular cylinder and sphere are studied by a method of successive approximations in which the first approximation gives the incompressible flow. Among such approximations is the Rayleigh–Janzen (RJ) method, sometimes called the M^2-expansion method, for dealing with steady flows of a compressible fluid when the flow field is irrotational and the maximum local Mach number is smaller than unity. An outline of this method and references are presented in Howarth (1953), Van Dyke (1964) and Imai (1957). Also, the accuracy of this method was investigated recently by Sakurai (1975).

The method applied here is similar to the RJ method. The fluid velocity is expanded in powers of ϵ, the ratio of the kinetic energy to the total energy of the fluid on the body surface; higher approximations are obtained by solving a set of ordinary differential equations with boundary conditions on the body surface. The results for a sphere up to the third approximation agree closely with Taylor's results.

2. Analysis

The equations of continuity and motion in polar co-ordinates are

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^j}\frac{\partial}{\partial r}(\rho v r^j) = 0, \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2}$$

where the parameter $j = 0$, 1 and 2 for one-, two- and three-dimensional flows, i.e. flows caused by a plane piston, a circular cylinder and a sphere respectively. In these equations, the starting point of the plane piston and the centres of the circular cylinder and sphere are at the origin. Since the flows are similarity ones, if we introduce a non-dimensional variable

$$r^* = r/u_p t, \tag{3}$$

where u_p is the velocity of the plane piston or the expansion velocity of the circular cylinder or sphere, the fluid velocity v, the density ρ and the pressure p are functions of r^* only. In other words, these quantities have the same values at points with the same value of r^* at different instants. In terms of r^*, (1) and (2) may be written respectively as

$$(u_p r^* - v)\frac{1}{\rho}\frac{d\rho}{dr^*} = \frac{dv}{dr^*}+\frac{jv}{r^*}, \tag{4}$$

$$\frac{1}{\rho}\frac{dp}{dr^*} = (u_p r^* - v)\frac{dv}{dr^*}. \tag{5}$$

If we assume that the flows are isentropic, ρ and p can be eliminated from (4) and (5) using the sound speed $a = (dp/d\rho)^{\frac{1}{2}}$ to give

$$\frac{dv}{dr^*}+\frac{jv}{r^*} = \frac{1}{a^2}(u_p r^* - v)^2\frac{dv}{dr^*}. \tag{6}$$

Integrating (5) with respect to r^* from the body surface to an arbitrary point yields

$$\frac{a^2}{\gamma-1}-\int_1^{r^*} u_p r^*\frac{dv}{dr^*}dr^* + \tfrac{1}{2}v^2 = \frac{a_p^2}{\gamma-1}+\tfrac{1}{2}u_p^2 = \tfrac{1}{2}C^2, \tag{7}$$

where a_p denotes the sound speed of the fluid on the body surface, γ is the ratio of specific heats and C is a constant with the dimensions of velocity. Eliminating a^2 from (6) and (7) and introducing a non-dimensional variable

$$\epsilon = u_p^2/C^2, \tag{8}$$

we get

$$(\gamma-1)\left\{1+2\epsilon\left(\int_1^{r^*}\frac{r^*}{u_p}\frac{dv}{dr^*}dr^* - \frac{1}{2}\frac{v^2}{u_p^2}\right)\right\}\left(\frac{dv}{dr^*}+\frac{jv}{r^*}\right) = 2\epsilon\left(r^*-\frac{v}{u_p}\right)^2\frac{dv}{dr^*}. \tag{9}$$

Here the variable ϵ is always smaller than unity.

Then we assume that the fluid velocity can be expanded in the form

$$v = v_0+\epsilon v_1+\epsilon^2 v_2 + \dots, \tag{10}$$

where v_0 denotes the fluid velocity in the case of incompressible fluid, because the condition of incompressibility corresponds to $a \to \infty$, i.e. $C \to \infty$, and consequently the condition that $\epsilon \to 0$. Substituting (10) into (9) and equating terms of the same order in ϵ we get a set of linear first-order ordinary differential equations:

$$\frac{dv_0}{dr^*}+\frac{jv_0}{r^*}=0, \tag{11a}$$

$$\frac{dv_1}{dr^*}+\frac{jv_1}{r^*}=\frac{2}{\gamma-1}\left(r^*-\frac{v_0}{u_p}\right)^2\frac{dv_0}{dr^*}, \tag{11b}$$

$$\begin{aligned}\frac{dv_2}{dr^*}+\frac{jv_2}{r^*} = & -2\left\{\left(\int_1^{r^*}\frac{r^*}{u_p}\frac{dv_0}{dr^*}-\frac{1}{2}\frac{v_0^2}{u_p^2}\right)\left(\frac{dv_1}{dr^*}+\frac{jv_1}{r^*}\right)\right\} \\ & +\frac{2}{\gamma-1}\left\{\left(r^*-\frac{v_0}{u_p}\right)^2\frac{dv_1}{dr^*}-\left(\frac{2v_1r^*}{u_p}-\frac{2v_0v_1}{u_p^2}\right)\frac{dv_0}{dr^*}\right\},\end{aligned} \tag{11c}$$

etc. The first term v_0 of (10) is determined by solving (11a) under the condition that $v_0 = u_p$ on the body surface, i.e. at $r^* = 1$; we obtain

$$v_0 = u_p/r^{*j}, \quad j = 0, 1, 2. \tag{12}$$

This shows that in the case of one-dimensional flow v_0 is constant, equal to the piston velocity, and that in two- and three-dimensional flow it gives the velocity field due to the corresponding point sources. The higher terms v_1, v_2, ..., can be obtained by solving the above equations successively under the condition that $v_1, v_2, \ldots = 0$ at $r^* = 1$. We find

$$v_1 = v_2 = \ldots \equiv 0 \tag{13a}$$

for one-dimensional flow,

$$\left.\begin{aligned} v_1 = & -\frac{u_p}{\gamma-1}\left(r^*-\frac{4\log r^*}{r^*}-\frac{1}{r^{*3}}\right), \\ v_2 = & -\frac{u_p}{(\gamma-1)^2}\left\{\frac{r^{*3}}{2}-6r^*+\frac{2}{r^*}+\frac{2}{r^{*3}}-\frac{5}{2r^{*5}}+4r^*\log r^*+\frac{20\log r^*}{r^*}\right. \\ & \left.-\frac{16(\log r^*)^2}{r^*}-\frac{12\log r^*}{r^{*3}}\right\}-\frac{u_p}{\gamma-1}\left\{-r^*+\frac{1}{2r^*}+\frac{1}{r^{*3}}-\frac{1}{2r^{*5}}\right. \\ & \left.+2r^*\log r^*+\frac{2\log r^*}{r^*}-\frac{2\log r^*}{r^{*3}}-\frac{4(\log r^*)^2}{r^*}\right\} \end{aligned}\right\} \tag{13b}$$

for two-dimensional flow and

$$\left.\begin{aligned} v_1 = & -\frac{u_p}{\gamma-1}\left(2-\frac{9}{r^{*2}}+\frac{8}{r^{*3}}-\frac{1}{r^{*6}}\right), \\ v_2 = & -\frac{u_p}{(\gamma-1)^2}\left\{18-\frac{32}{r^*}-\frac{513}{10r^{*2}}+\frac{144}{r^{*3}}-\frac{78}{r^{*4}}-\frac{27}{r^{*6}}+\frac{144}{5r^{*7}}-\frac{5}{2r^{*10}}\right\} \\ & -\frac{u_p}{\gamma-1}\left\{8-\frac{16}{r^*}-\frac{63}{10r^{*2}}+\frac{32}{r^{*3}}-\frac{18}{r^{*4}}-\frac{4}{r^{*6}}+\frac{24}{5r^{*7}}-\frac{1}{2r^{*10}}\right\} \end{aligned}\right\} \tag{13c}$$

for three-dimensional flow.

It follows from these results that the fluid velocity for one-dimensional flow is equal to u_p independently of the value of ϵ. This corresponds the well-known flow in which a shock wave exists in front of the piston and the fluid ahead of the shock wave is at rest. We do not treat this flow further in this paper. For two- and three-dimensional flow, both v_1 and v_2 are negative monotonic-decreasing functions of r^*, and when $r^* \to \infty$, v_1 and v_2 tend to $-\infty$ for two-dimensional flow and asymptotically approach finite negative values for three-dimensional flow. The same tendency is expected for the other higher terms $v_3, v_4, \ldots$. Since the lowest-order fluid velocity v_0 approaches zero asymptotically as $r^* \to \infty$, we find that the resultant fluid velocity v decreases monotonically to a negative value as r^* increases when ϵ is not zero.

However, since a shock wave is formed around an expanding body with a finite velocity, the shock wave can be considered to appear before v falls below zero. The position r_s^* of the shock wave may be found from a geometrical condition (the ratio of the velocities of the shock wave and the body is the same as that of their radii) and the normal shock condition, i.e. r_s^* can be obtained by eliminating M_s from the two equations

$$M_s = u_p/a_\infty r_s^*, \tag{14}$$

$$v_s/a_\infty = 2(M_s^2-1)/(\gamma+1)\,M_s. \tag{15}$$

Here M_s (>1) denotes the shock Mach number, the ratio of the shock velocity to the sound speed a_∞ in the undisturbed region, and v_s denotes the fluid velocity just behind the shock wave.

3. Calculation and results

Since the quantity C in (7) is constant throughout the area between the shock wave and the body, it can be obtained from the condition just behind the shock wave:

$$\frac{a_s^2}{\gamma-1} - \int_1^{r_s^*} u_p\, r^* \frac{dv}{dr^*}\, dr^* + \tfrac{1}{2}v_s^2 = \tfrac{1}{2}C^2. \tag{16}$$

Here the sound speed in the fluid just behind the shock wave is related to that in the undisturbed region by the following relation for a normal shock wave:

$$\frac{a_s}{a_\infty} = \left\{1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{\gamma M_s^2+1}{M_s^2}(M_s^2-1)\right\}^{\frac{1}{2}}. \tag{17}$$

In order to determine v_s in (16), however, the value of ϵ, i.e. C, must be known beforehand, as is obvious from (10). Conversely, in order to obtain the constant C, it is necessary to know the position of the shock wave, and consequently the value of v_s. These cannot be obtained by solving the equations. Therefore we first find a velocity distribution from (10) by taking an arbitrary value of ϵ and determine the corresponding position of the shock wave; then, using (16) and (17), we calculate the constant C, from which we get another value of ϵ and again substitute it into (10). This process is repeated until the calculated values converge.

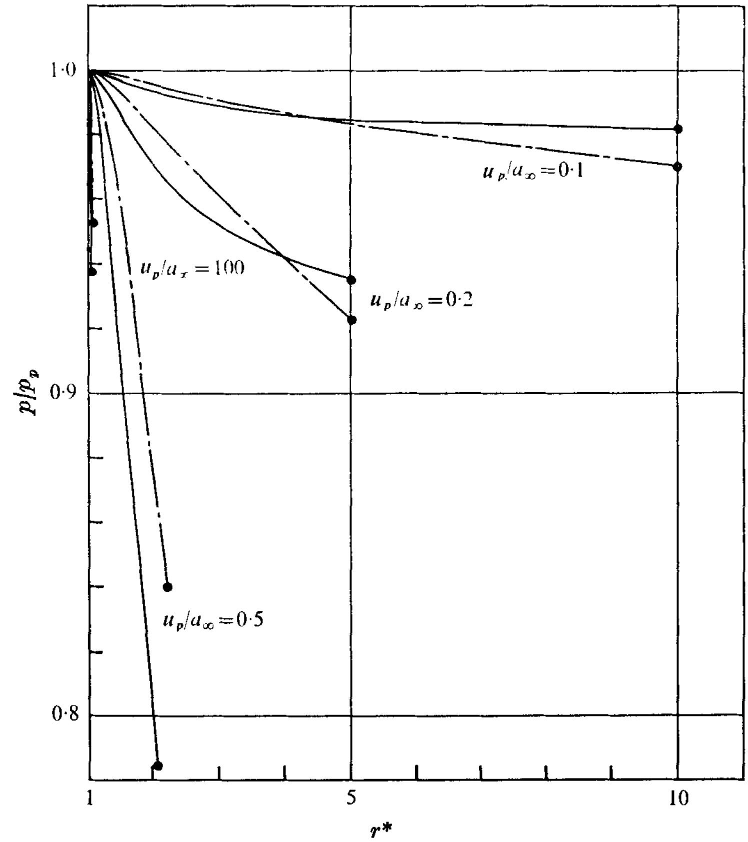

FIGURE 1. Pressure distributions. — - —, circular cylinder; ————, sphere.

The pressure distribution is obtained from the distribution of sound speed, which is given by (7), through the isentropic relation

$$\frac{p}{p_p} = \left(\frac{a}{a_p}\right)^{2\gamma/(\gamma-1)}, \tag{18}$$

where the suffix p refers to the value on the body surface. The pressure distributions for a circular cylinder and a sphere are shown in figure 1.

A comparison between the present results (γ is taken as 1·405 for this comparison) and those of Taylor for the position of the shock wave and the pressure for a sphere is given in table 1. The values of ϵ are also presented for interest. This shows that the present results agree with Taylor's; in particular when the expansion rate is very large ($u_p/a_\infty > 100$) the shock position r_s^* and the ratio p_s/p_p found by our method are almost constant, 1·062 and 0·937, respectively, compared with Taylor's results 1·060 and 0·93 for $u_p/a_\infty \to \infty$. We consider that the error due to neglecting terms higher than v_2, which is $O(\epsilon^3)$ in the present approximation, becomes significant as r^* increases because $|v_3|$, $|v_4|$, ... increase monotonically with r^*; however when ϵ is large, i.e. when u_p/a_∞ is rather large, r_s^* is small because the shock wave approaches the body. Consequently the error is expected to be small for all cases.

		Position of shock wave, r_s^*		Pressure, p_s/p_p	
u_p/a_∞	ϵ	Present analysis	Taylor	Present analysis	Taylor
0·203	0·00811	4·928	4·93	0·934	0·928
0·523	0·04773	1·992	1·950	0·785	0·750
1·180	0·16793	1·263	1·256	0·815	0·811
3·598	0·39398	1·085	1·083	0·917	0·92
10·0	0·46059	1·065	—	0·934	—
100·0	0·47244	1·062	—	0·937	—
1000·0	0·47244	1·062	—	0·937	—
∞	—	—	1·060	—	0·93

TABLE 1. Comparison with Taylor's results on shock position and pressure for a sphere.

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